Recall from slide 2, lecture 6:

$$\boxed{\left(\frac{d\sigma(\theta)}{d\Omega}\right) = \frac{2\pi}{\hbar}\,\frac{V_n}{c\, d\Omega}\,\left|M_{if}\right|^2 \rho_f}$$

$$\begin{cases} M_{if} = \int \psi_f^* \, V(\vec{r})\, \psi_i \, d^3 r \\ \rho_f = dn / dE_F \end{cases}$$

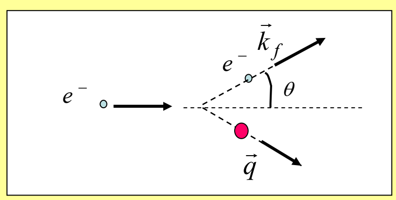
for the cross-section:

$$\left(\frac{d\sigma(\theta)}{d\Omega}\right) = \frac{2\pi}{\hbar}\,\frac{1}{c\, V_n}\left(\frac{Z e^2}{4\pi\,\varepsilon_o}\right)^2 \left(\frac{4\pi}{q^2 + \alpha^{-2}}\right)^2 \left(F(q^2)\right)^2 \boxed{\frac{dn}{dE_F\, d\Omega}}$$

All we have left to calculate is the "density of states" factor, $\frac{dn}{dE_F\, d\Omega}$ where E_F is the total energy in the final state when the electron scatters at angle θ, and this factor accounts for the number of ways it can do that.

Consider the **total** final state energy:

$$\frac{dn}{dE_F\, d\Omega} = \frac{dn}{dp_f\, d\Omega}\left(\frac{dp_f}{dE_F}\right)_\theta$$



$$E_F = \underbrace{E'}_{\text{(electron)}} + \underbrace{E_R}_{\text{(recoil)}}$$

$$E_F = (cp_f + mc^2) + (Mc^2 + K)$$ *(being careful with the factor of c !)*

$$\frac{dp_f}{dE_F} = \frac{W\, p_f}{Mc^3\, p_i} \approx \frac{1}{c}$$

$$dE_F \cong c\, dp_f$$

$$\frac{dn}{dE_F\, d\Omega} = \frac{dn}{c dp_f\, d\Omega}$$

This is useful because the momentum states are quantized – we have our electrons in a normalization volume, and we can "count the states" inside ...

Counting the scattered electron momentum states: 3

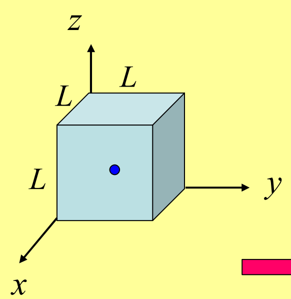
Recall the wave function:

$$\psi_f(\vec{r}) = \frac{1}{\sqrt{V_n}}\, e^{i\vec{k}_f \bullet \vec{r}} \quad with \quad p_f = \hbar k_f$$

The normalization volume is arbitrary, but we have to be consistent ....

let $V_n = L^3$, i.e. the electron wave function is contained in a cubical box.

Use periodic boundary conditions – wave function is the same on all sides of the box



Since: $\vec{k}_f \bullet \vec{r} \equiv k_x x + k_y y + k_z z$

Then it follows that:

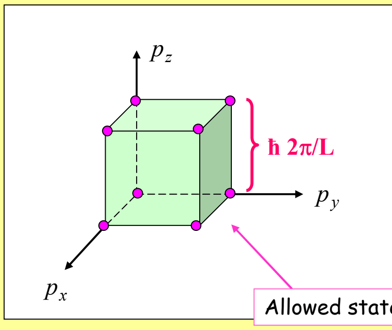
$$\psi_f(x,y,z) = \frac{1}{\sqrt{L^3}}\, e^{ik_x x}\, e^{ik_y y}\, e^{ik_z z}$$

$$\boxed{k_x L = n_x\ 2\pi,\ \ etc. \ldots}$$

So, momentum is quantized on a 3-d lattice: 4

$$\vec{p}_f = \hbar\,\vec{k}_f = \hbar\left(\frac{2\pi}{L}\right)\left(n_x\,\hat{i} + n_y\,\hat{j} + n_z\,\hat{k}\right)$$

$$n_x = \pm(1, 2, 3\ldots) \quad etc.$$



For a relativistic electron beam, the quantum numbers n_x *etc.* are very large, but finite.

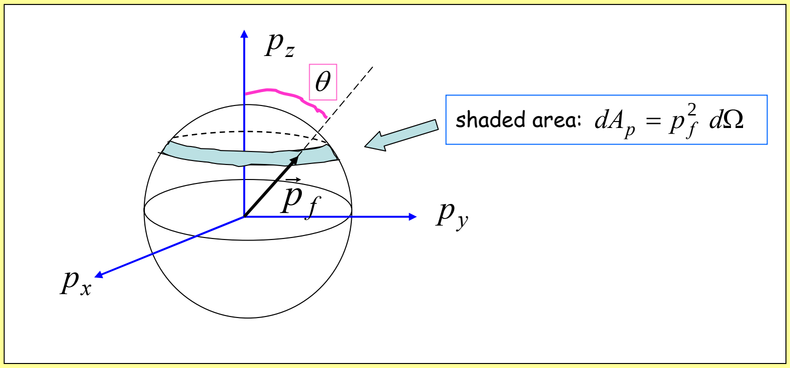
We use the quantization relation **not** to calculate the allowed momentum, but rather to calculate **the density of states!**

Allowed states are dots, 1 per cube of volume $\tau_p = (2\pi\hbar/L)^3$

$$\frac{dn}{d\tau_p} = \frac{1\ \text{state}}{(2\pi\,\hbar/L)^3}$$

See F&H section 10.2 ...

Finally, consider the scattered momentum into dΩ at θ : 5



number of momentum points in the shaded ring: $dn = \left(\frac{dn}{d\tau_p}\right) \times \left(dA_p \; dp_f\right)$

$$\Rightarrow \quad dn = \frac{V_n}{(2\pi \hbar)^3} \; p_f^2 \; dp_f \; d\Omega$$

End of the calculation: 6

$$dn = \frac{V_n}{(2\pi \hbar)^3} \; p_f^2 \; dp_f \; d\Omega$$

We want the density of states factor:

$$\boxed{\frac{dn}{dE_F \; d\Omega} = \frac{dn}{cdp_f \; d\Omega} = \frac{V_n}{(2\pi \hbar)^3} \; \frac{p_f^2}{c}}$$

FINALLY, from slide 10:

$$\left(\frac{d\sigma(\theta)}{d\Omega}\right) = \frac{2\pi}{\hbar} \; \frac{1}{c\,\cancel{V_n}} \left(\frac{Z\,e^2}{4\pi\,\varepsilon_o}\right)^2 \left(\frac{4\pi}{q^2 + \alpha^{-2}}\right)^2 \left(F(q^2)\right)^2 \boxed{\left(\frac{\cancel{V_n}}{(2\pi \hbar)^3} \; \frac{p_f^2}{c}\right)}$$

$$= \left(\text{point charge cross-section}\right) \times \left(F(q^2)\right)^2$$

Result: Cross section for electron scattering from nuclear charge Z: 7

$$\left(\frac{d\sigma(\theta)}{d\Omega}\right) = \frac{4\,Z^2}{\hbar^2\,(\hbar c)^2}\left(\frac{e^2}{4\pi\,\varepsilon_o}\right)^2 \frac{p_f^2}{(q^2+\alpha^{-2})^2}\left(F(q^2)\right)^2$$

$$\cong \boxed{\frac{4\,Z^2}{(\hbar c)^4}\left(\frac{e^2}{4\pi\,\varepsilon_o}\right)^2 \frac{(cp_f)^2}{q^4}}\ \boxed{\left(F(q^2)\right)^2}$$

point charge cross-section:
most notably, falls off as q^{-4}
(units should be fm^2)

form factor squared
(dimensionless)

Check units: $[\hbar c] = [e^2/4\pi\varepsilon_o] = \text{MeV.fm};\ [cp] = \text{MeV};\ [q] = \text{fm}^{-1}$

$$\left[\frac{d\sigma}{d\Omega}\right] = \frac{1}{(\text{MeV.fm})^4}(\text{MeV.fm})^2\,\frac{(\text{MeV})^2}{\text{fm}^{-4}} = \text{fm}^2$$

What does this function look like? 8

$$\left(\frac{d\sigma(\theta)}{d\Omega}\right) = \boxed{\frac{4\,Z^2}{(\hbar c)^4}\left(\frac{e^2}{4\pi\,\varepsilon_o}\right)^2}\ \frac{(cp_f)^2}{q^4}\left(F(q^2)\right)^2 = Z^2\boxed{\left(\frac{d\sigma(\theta)}{d\Omega}\right)_o}\left(F(q^2)\right)^2$$

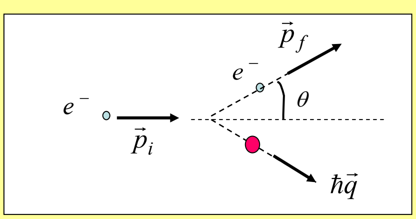
First calculate the point charge cross section for Z = 1:

point charge

constants:

$$\frac{4}{(\hbar c)^4}\left[\frac{e^2}{4\pi\,\varepsilon_o}\right]^2 = \frac{4}{(197.5)^4}[1.44]^2 = 5.45\times 10^{-9}\ (\text{MeV.fm})^{-2}$$

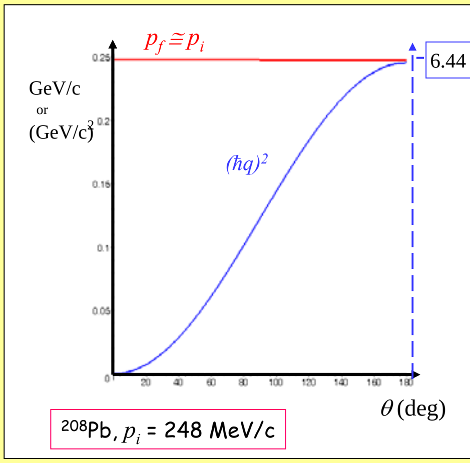
For the kinematic factors, we need our results from lecture 5 for the momenta, being of course very careful with the units:



$$p_f = \frac{p_i}{1+\frac{p_i}{M}(1-\cos\theta)}$$

$$(\hbar q)^2 = p_i^2 + (p_f)^2 - 2p_i\,p_f\cos\theta$$

First, let's consider a heavy nuclear target, so that $p_f \cong p_i$ and work out $Z^2 d\sigma/d\Omega)_o$
Choose an example of 248 MeV/c electrons scattering from ^{208}Pb and compare to data.



Kinematic relations become:

$$p_f \cong p_i = \text{constant}$$

$$(\hbar q)^2 \cong 2\, p_i^2\, (1-\cos\theta)$$

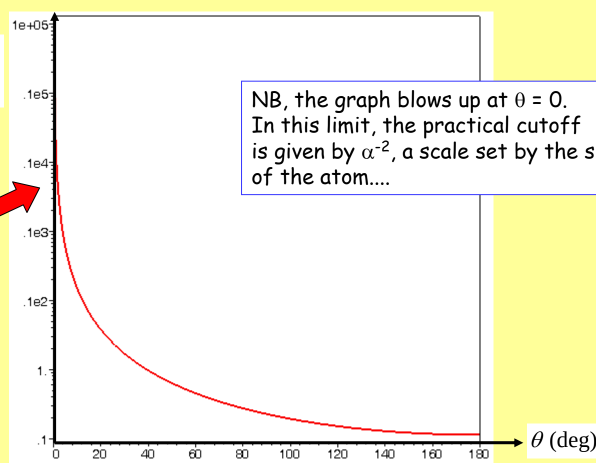
$$\frac{d\sigma}{d\Omega} \sim \frac{p_f^2}{q^4}$$

$$1-\cos\theta = 2\sin^2(\theta/2)\ ...$$

$$\left(\frac{d\sigma}{d\Omega}\right)_o = \frac{const.}{q^4} \sim \frac{1}{\sin^4(\theta/2)}$$

$$Z^2\left(\frac{d\sigma}{d\Omega}\right)_o = \frac{const.}{q^4} = \frac{0.0567\ \text{fm}^2/\text{sr}}{\sin^4(\theta/2)}$$

$$Z^2\left(\frac{d\sigma}{d\Omega}\right)_o \ \text{fm}^2/\text{sr}$$



NB, the graph blows up at $\theta = 0$.
In this limit, the practical cutoff is given by α^{-2}, a scale set by the size of the atom....

Log scale! The point charge cross section drops
like a stone!!!

How are we doing? Compare to data: (See F&H sec. 6.5) 11

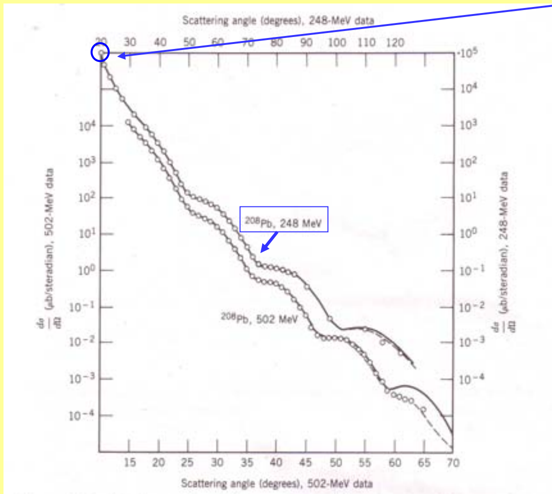


Figure 3.2 Elastic scattering of electrons from ^{208}Pb. Note the different vertical and horizontal scales for the two energies. This also shows diffractionlike behavior, but lacks sharp minima. (J. Heisenberg et al., *Phys. Rev. Lett.* **23**, 1402 (1969).)

$\theta = 10°$,
$d\sigma/d\Omega = 10^5$ μb/sr
$= 10$ fm^2/sr ...

Point charge calculation:
$d\sigma(10°)/d\Omega = 982$ fm^2/sr

$$\frac{d\sigma}{d\Omega} = Z^2 \left(\frac{d\sigma}{d\Omega}\right)_o \left(F(q^2)\right)^2$$

$(F(q^2))^2$ at $10°$ =
$10/982 = 0.01$!!!

(The correction from relativistic quantum mechanics gives a $\cos^2(\frac{1}{2}\theta)$ multiplicative correction, which is trivial here.)

Also, the graphs are not smooth like $1/q^4$ -- evidence that the target has finite size!

Summary so far: 12

We can predict the cross-section exactly for a pointlike target with nonrelativistic quantum mechanics.

(This approach is correct for a target particle that has charge but no magnetic moment, i.e. intrinsic angular momentum of **zero**. We can't use this for the proton without adding some refinements, so along the way we are stopping to look at the charge distributions of nuclei. Nuclei with (Z, N) both **even**, such as ^{208}Pb, have angular momentum zero, so our theory is perfect for this case!)

Recall our basic result:

$$\left(\frac{d\sigma(\theta)}{d\Omega}\right) = (\text{point charge cross-section}) \times \left(F(q^2)\right)^2$$

Where $F(q^2)$ is the Fourier transform of the target charge density:

$$F(q^2) \equiv \int e^{i\vec{q}\cdot\vec{r}} \rho(r)\, d^3r$$

Measuring $d\sigma/d\Omega$ and dividing by the point charge result yields a value for $F(q^2)$

How to find the charge distribution? 13

form factor: $F(q^2) \equiv \int e^{i\vec{q}\cdot\vec{r}} \rho(r)\, d^3r$

inverse Fourier transform: $\rho(r) \equiv \frac{1}{(2\pi)^3} \int e^{-i\vec{q}\cdot\vec{r}} F(q^2)\, d^3q$

In principle, one could measure the form factor, and numerically integrate to invert the Fourier transform and find $\rho(r)$.

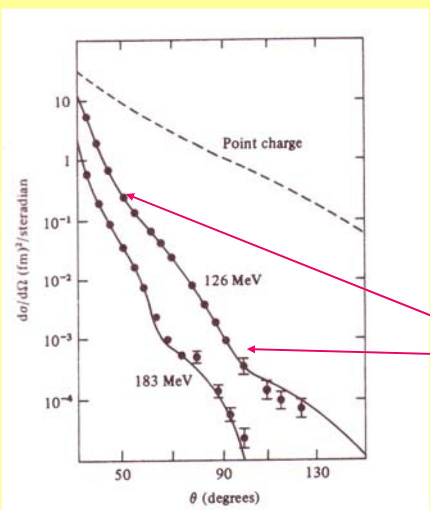
However, this doesn't work in practice, because the integral has to be done over a complete range of q from 0 to ∞, and no experiment can ever span an infinite range of momentum transfer!

(It is bad enough trying to acquire data at large momentum transfer because the basic cross-section drops like q^{-4} → the rate of scattered particles into a detector gets too small – see lecture 4)

What to do? ...

A practical solution: 14

Experimental data are fitted to a functional form for $F(q^2)$; parameters extracted from the fit are used to invert the transform and deduce $\rho(r)$....



Example: elastic electron scattering from gold (A=197, Z = 79). Best fit to data is given by solid curves.

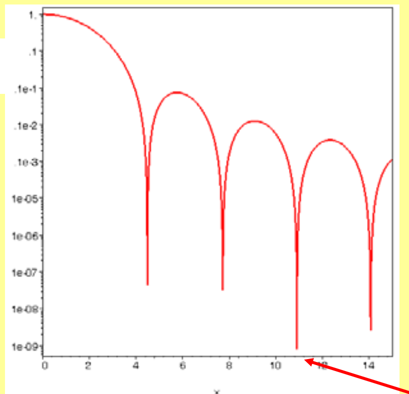
(Ref: R. Hofstadter, *Electron Scattering & Nuclear Structure*, 1963)

Discontinuities are evidence of diffraction-like behavior, characteristic of a Fourier transform, but the edges are fuzzy!

Illustration of "Diffraction" behavior in F(q²): Uniform sphere

$$F(q^2) \equiv \int e^{i\vec{q}\cdot\vec{r}} \rho(r)\, d^3r = \int_0^{2\pi} d\phi \int_0^{\infty} r^2 \rho(r) dr \int_0^{\pi} \sin\theta\, e^{iqr\cos\theta}\, d\theta$$

$(F(q^2))^2$



Density:

$\rho = \rho_o, \quad r < R;$

$\rho = 0, \quad r > R$

$$F(q^2) = \frac{3(\sin x - x\cos x)}{x^3}$$

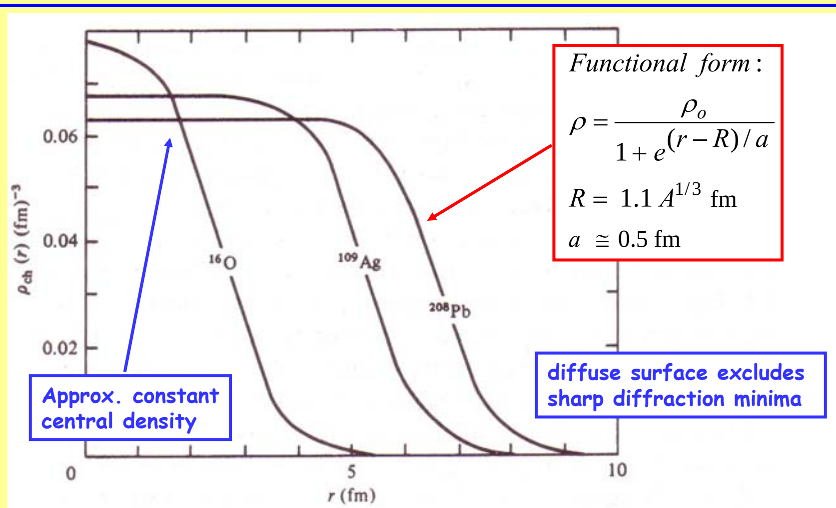
$\{ x = qR \}$

Important scaling property – in dimensionless variable (qR)

Zeroes never quite get to zero on a log scale!

In contrast, the **minima are not as sharp for nuclei**…

Nuclear charge distributions from experiment:



Functional form:

$$\rho = \frac{\rho_o}{1 + e^{(r-R)/a}}$$

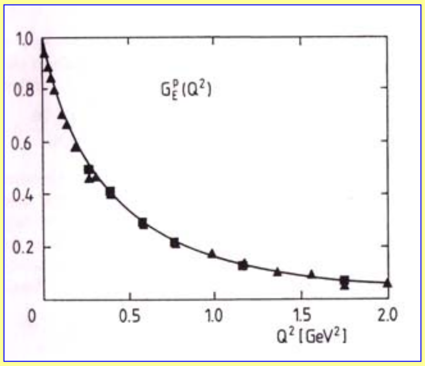
$R = 1.1\, A^{1/3}$ fm

$a \cong 0.5$ fm

Fig. 4.3 The electric charge density of three nuclei as fitted by $\rho_{ch}(r) = \rho^0_{ch}/[1 + \exp((r - R)/a)]$. The parameters are taken from the compilation in Barrett, R. C. & Jackson, D. F. (1977), *Nuclear Sizes and Structure*, Oxford: Clarendon Press.

Contrast to the proton (recall lecture 4): 17

Proton Form factor data:



$$G_E^p(Q^2) \;=\; \left(1+\frac{Q^2}{0.71\,\mathrm{GeV}^2}\right)^{-2}$$

Electric charge distribution:

$$\rho(r) = e\rho_o \exp(-M\,r)$$

$$M \;=\; 4.33\;\; \mathrm{fm}^{-1}$$

$$\left\langle r^2 \right\rangle^{1/2} \;=\; \frac{\sqrt{12}}{M} \;=\; 0.80\;\; \mathrm{fm}$$

$\dfrac{\rho}{e}\;\; \mathrm{fm}^{-3}$